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A Mathematical Note on the Feynman Path Integral for the Quantum Electrodynamics

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1 Introduction.

A number of mathematical results on the Feynman path integral for the quantum mechanics have been obtained. On the other hand, the author doesn't know any mathematical results on the Feynman path integral for the quantum electrodynamics, written as QED from now on. A functional integral representation for a non-relativistic QED model with imaginary time was obtained by Hiroshima (1997) [7] by means of the probabilistic method.

Our aim in the present paper is to give the mathematical definition of the Feynman path integral for the non-relativistic QED, especially studied in Feynman (1950) [4] and Feynman - Hibbs (1965) [5]. In the present paper the Fourier series is used as in Fermi (1932) [2], Feynman [4] and Sakurai (1967) [13], and photons with large momentum are arbitrarily cut off.

We first give the mathematical definition of the Feynman path integral for the non-relativistic QED under the constraint condition, whose method is well

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known (cf. (9.7) in [5], (A-7) in [13], (13.10) in Spohn (2004) [14] and (7.38) in Swanson (1992) [15]). Secondly, without the constraint condition we give the mathematical definition of the Feynman path integral for the non-relativistic QED, which is given by (9-98) in [5]. The author emphasize that any concrete definition of (9-98) in [5] is not given. So our result may be completely new. We also note that our Feynman path integral without the constraint condition is proved to be equal with the Feynman path integral under the constraint condition.

Our plan in the present paper is as follows. §2 is devoted to preliminaries. In §3 we give the mathematical definition of the Feynman path integral for the non-relativistic QED under the constraint condition and prove that this Feynman path integral converges. We also state some remarks. In particular, the expressions of the Hamiltonian operator and etc. by means of the creation operators and the annihilation operators are given in Remark 3.3. These expressions are used in many literatures (cf. Gustafson-Sigal (2003) [6], [7], [13] and [14]). In §4 we give the mathematical definition of the Feynman path integral for the non-relativistic QED without the constraint condition and prove that our Feynman path integral without the constraint condition is equal to the Feynman path integral under the constraint condition. So, the Feynman path integral without the constraint condition is also proved to converge from the result in §3.

2 Preliminaries.

We consider n charged non-relativistic particles $x^{(j)} \in R^3$ $(j = 1, 2, \ldots, n)$ with mass $m_j > 0$ and charge $e_j \in R$. Let $t \in [0, T], \phi(t, x) \in R$ a scalar potential and $A(t, x) \in R^3$ a vector potential, respectively. We set

$$\vec{x} := (x^{(1)}, \ldots, x^{(n)}) \in R^{3n},$$

$$\dot{\vec{x}} := (\dot{x}^{(1)}, \ldots, \dot{x}^{(n)}) \in R^{3n}.$$

Then the Lagrangian function for the particles and the electromagnetic field with

$$\rho(t, x) = \sum_{j=1}^{n} e_j \delta\left(x - x^{(j)}(t)\right) \tag{2.1}$$

and

$$j(t, x) = \sum_{j=1}^{n} e_j \dot{x}^{(j)}(t) \delta\left(x - x^{(j)}(t)\right) \in R^3 \tag{2.2}$$

is given by

$$\begin{aligned}
&\mathcal{L}\left(t, \vec{x}, \dot{\vec{x}}, A, \dot{A}, \frac{\partial A}{\partial x}, \phi, \frac{\partial \phi}{\partial x}\right) \\
&= \sum_{j=1}^{n} \left(\frac{m_j}{2}|\dot{x}^{(j)}|^2 - \int \rho(t, x)\phi(t, x)dx + \frac{1}{c}\int j(t, x) \cdot A(t, x)dx\right) \\
&+ \frac{1}{8\pi}\int_{R^3} \left(|E(t, x)|^2 - |B(t, x)|^2\right) dx + \text{Const.} \\
&= \sum_{j=1}^{n} \left(\frac{m_j}{2}|\dot{x}^{(j)}|^2 - e_j\phi(t, x^{(j)}) + \frac{1}{c}e_j\dot{x}^{(j)} \cdot A(t, x^{(j)})\right) \\
&+ \frac{1}{8\pi}\int_{R^3} \left(|E(t, x)|^2 - |B(t, x)|^2\right) dx + \text{Const.}
\end{aligned} \tag{2.3}$$

(cf. [5], [14]), where

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial \phi}{\partial x}, \quad B = \nabla \times A \tag{2.4}$$

and we note that the Lagrangian function (2.3) has an arbitrary constant.

As in Fermi [2], Feynman [4] and Sakurai [13] we consider a sufficient large box

$$V = \left[-\frac{L_1}{2}, \frac{L_1}{2}\right] \times \left[-\frac{L_2}{2}, \frac{L_2}{2}\right] \times \left[-\frac{L_3}{2}, \frac{L_3}{2}\right]$$

and periodic potentials $\phi(t,x)$ and $A(t,x)$ such that

$$\nabla \cdot A(t,x) = 0 \quad \text{in } [0,T] \times R^3 \quad \text{(the Coulomb gauge)} \tag{2.5}$$

and

$$\int_V \phi(t,x)dx = 0, \qquad \int_V A(t,x)dx = 0. \tag{2.6}$$

Let $|V| = L_1L_2L_3$. We set

$$k := \left(\frac{2\pi}{L_1}s_1, \frac{2\pi}{L_2}s_2, \frac{2\pi}{L_3}s_3\right) \quad (s_1, s_2, s_3 \in \mathbf{Z}) \tag{2.7}$$

and take $\vec{e_j}(k) \in R^3$ $(j = 1,2)$ such that $(\vec{e_1}(k), \vec{e_2}(k), k/|k|)$ for all $k \neq 0$ forms a set of mutually orthogonal unit vectors and

$$\vec{e_j}(-k) = -\vec{e_j}(k) \quad (j = 1,2) \tag{2.8}$$

(cf. p. 448 in Arai 2000 [1]). Then we can expand $\phi(t,x)$ and $A(t,x)$ from (2.5) and (2.6) into plane waves

$$A(x, \{a_{lk}\}) = \frac{\sqrt{4\pi}}{|V|}c\sum_{k\neq 0}\{a_{1k}e^{ik\cdot x}\vec{e_1}(k) + a_{2k}e^{ik\cdot x}\vec{e_2}(k)\}, \tag{2.9}$$

$$\phi(x, \{a_{lk}\}) = \frac{1}{|V|}\sum_{k\neq 0}\phi_k e^{ik\cdot x}. \tag{2.10}$$

We write

$$a_{lk} =: \frac{a_{lk}^{(1)} - ia_{lk}^{(2)}}{\sqrt{2}} \ (l = 1,2), \tag{2.11}$$

$$\phi_k =: \phi_k^{(1)} - i\phi_k^{(2)}. \tag{2.12}$$

Since A and ϕ are real valued, the relations

$$a_{\mathrm{l}-k}^{(1)} = -a_{\mathrm{l}k}^{(1)}, \quad a_{\mathrm{l}-k}^{(2)} = a_{\mathrm{l}k}^{(2)}, \quad \phi_{-k}^{(1)} = \phi_k^{(1)}, \quad \phi_{-k}^{(2)} = -\phi_k^{(2)} \tag{2.13}$$

hold from (2.8). So, we have

$$A(x, \{a_{\mathrm{l}k}\}) = \frac{\sqrt{\pi}}{|V|} c \sum_{k\neq 0} \sum_{\mathrm{l}=1}^{2} \left(a_{\mathrm{l}k} e^{ik\cdot x} + a_{\mathrm{l}k}^{*} e^{-ik\cdot x}\right) \vec{e_{\mathrm{l}}}(k)$$

$$= \frac{\sqrt{4\pi}}{|V|} c \sum_{k\neq 0} \sum_{\mathrm{l}=1}^{2} \frac{1}{\sqrt{2}} \left(a_{\mathrm{l}k}^{(1)} \cos k\cdot x + a_{\mathrm{l}k}^{(2)} \sin k\cdot x\right) \vec{e_{\mathrm{l}}}(k), \tag{2.14}$$

$$\phi(x, \{a_{\mathrm{l}k}\}) = \frac{1}{|V|} \sum_{k\neq 0} \left(\phi_k^{(1)} \cos k\cdot x + \phi_k^{(2)} \sin k\cdot x\right), \tag{2.15}$$

where $a_{\mathrm{l}k}^{*}$ denotes the complex conjugate of $a_{\mathrm{l}k}$. We also write

$$\rho_k^{(1)}(\vec{x}) := \sum_{j=1}^{n} e_j \cos k\cdot x^{(j)}, \tag{2.16}$$

$$\rho_k^{(2)}(\vec{x}) := \sum_{j=1}^{n} e_j \sin k\cdot x^{(j)}. \tag{2.17}$$

Determining an arbitrary constant in the Lagrangian function (2.3) as follows, we define $\mathcal{L}$ by

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}, \{a_{\mathrm{l}k}\}, \{\dot{a}_{\mathrm{l}k}\}, \{\phi_k\}) = \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2$$

$$+ \frac{1}{8\pi|V|} \sum_{k\neq 0} \left\{ \sum_{\mathrm{i}=1}^{2} \left(|k|^2 |\phi_k^{(\mathrm{i})}|^2 - 8\pi \rho_k^{(\mathrm{i})}(\vec{x})\ \phi_k^{(\mathrm{i})} \right) \right.$$

$$\left. + 16\pi^2 \frac{\sum_{j=1}^{n} e_j^2}{|k|^2} \right\} + \frac{1}{c} \sum_{j=1}^{n} e_j\ \dot{x}^{(j)} \cdot A(x^{(j)}, \{a_{\mathrm{l}k}\})$$

$$+ \frac{1}{2} \sum_{k\neq 0,\mathrm{i},\mathrm{l}} \left(\frac{|\dot{a}_{\mathrm{l}k}^{(\mathrm{i})}|^2}{2|V|} - \frac{(c|k|)^2 |a_{\mathrm{l}k}^{(\mathrm{i})}|^2}{2|V|} + \frac{hc|k|}{2} \right). \tag{2.18}$$

Taking account of the constraint condition

$$|k|^2\phi_k^{(\mathrm{i})} = 4\pi\rho_k^{(\mathrm{i})} \quad (\mathrm{i}=1,2,\ k\neq 0), \tag{2.19}$$

roughly $\nabla \cdot E = 4\pi\rho$ (cf. (9-17) in [5] and (7.38) in [15]), then we have

$$\begin{aligned}
\mathcal{L}_c(\vec{x}, \dot{\vec{x}}, \{a_{\mathrm{l}k}\}, \{\dot{a}_{\mathrm{l}k}\}) &= \sum_{j=1}^{n} \frac{m_j}{2}|\dot{x}^{(j)}|^2 \\
&- \frac{2\pi}{|V|}\sum_{j\neq l}^{n}\sum_{k\neq 0} \frac{e_j e_l \cos k\cdot(x^{(j)}-x^{(l)})}{|k|^2} \\
&+ \frac{1}{c}\sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot A(x^{(j)}, \{a_{\mathrm{l}k}\}) \\
&+ \frac{1}{2}\sum_{k\neq 0,\mathrm{i},\mathrm{l}} \left(\frac{|\dot{a}_{\mathrm{l}k}^{(\mathrm{i})}|^2}{2|V|} - \frac{(c|k|)^2|a_{\mathrm{l}k}^{(\mathrm{i})}|^2}{2|V|} + \frac{\hbar c|k|}{2} \right).
\end{aligned} \tag{2.20}$$

3 Results under the constraint condition.

We arbitrarily cut off the terms of large wave numbers k in (2.20). That is, let M_j $(j=1,2,3)$ be arbitrary positive integers such that $M_2 \le M_3$. We consider

$$\begin{aligned}
\Lambda_j := \Big\{ k = \left(\frac{2\pi}{L_1}s_1, \frac{2\pi}{L_2}s_2, \frac{2\pi}{L_3}s_3\right); s_1^2+s_2^2+s_3^2 \neq 0, \\
|s_1|, |s_2|, |s_3| \le M_j \Big\}
\end{aligned} \tag{3.1}$$

and write

$$\Lambda_j =: \Lambda_j' \cup -\Lambda_j',\ \ \Lambda_j' \cap -\Lambda_j' = \text{empty set},\ \ \Lambda_2' \subseteq \Lambda_3'. \tag{3.2}$$

Let N_j denotes the number of elements of the set Λ_j'. It follows from (2.13) that independent variables are $a_{\Lambda_j'} := \{a_{\mathrm{l}k}^{(\mathrm{i})}\}_{k\in\Lambda',\mathrm{i},\mathrm{l}} \in R^{4N_j}$ (cf. p. 154 in [14]).

We consider

$$\tilde{\mathcal{L}}_c(\vec{x}, \dot{\vec{x}}, \{a_{lk}\}, \{\dot{a}_{lk}\}) := \sum_{j=1}^{n} \frac{m_j}{2}|\dot{x}^{(j)}|^2$$

$$- \frac{2\pi}{|V|} \sum_{j \neq l}^{n} \sum_{k \in \Lambda_1} \frac{e_j e_l \cos k \cdot (x^{(j)} - x^{(l)})}{|k|^2}$$

$$+ \frac{1}{c} \sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot \tilde{A}(x^{(j)}, \{a_{lk}\})$$

$$+ \frac{1}{2} \sum_{k \in \Lambda_3, \mathrm{i}, \mathrm{l}} \left(\frac{|\dot{a}_{\mathrm{l}k}^{(\mathrm{i})}|^2}{2|V|} - \frac{(c|k|)^2 |a_{\mathrm{l}k}^{(\mathrm{i})}|^2}{2|V|} + \frac{hc|k|}{2} \right) \tag{3.3}$$

in place of $\mathcal{L}_c$, where A given by (2.14) is replaced with

$$\tilde{A}(x, \{a_{lk}\}) = \frac{\sqrt{4\pi}}{|V|} cg(x) \sum_{k \in \Lambda_2} \sum_{\mathrm{l}=1}^{2} \Big(\psi(a_{\mathrm{l}k}^{(1)}/\sqrt{2}) \cos k \cdot x$$

$$+ \psi(a_{\mathrm{l}k}^{(2)}/\sqrt{2}) \sin k \cdot x \Big) \vec{e_{\mathrm{l}}}(k). \tag{3.4}$$

We suppose $\psi(-\theta) = -\psi(\theta)$ $(\theta \in R)$. We note that if $g = 1$ and $\psi(\theta) = \theta$, then $\tilde{A} = A$.

For the sake of simplicity we suppose $\Lambda' := \Lambda'_1 = \Lambda'_2 = \Lambda'_3$. Let

$$\Delta : 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu = T, \quad |\Delta| := \max_{1 \le l \le \nu} (\tau_l - \tau_{l-1}).$$

Let $\vec{x} \in R^{3n}$ and $a_{\Lambda'} \in R^{4N}$ $(N := N_1)$ be fixed. We take arbitrarily

$$\vec{x}^{(0)}, \ldots, \vec{x}^{(\nu-1)} \in R^{3n}$$

and

$$a_{\Lambda'}^{(0)}, \ldots, a_{\Lambda'}^{(\nu-1)} \in R^{4N}.$$

Then, we write the broken line paths on $[0, T]$ connecting $\vec{x}^{(l)}$ at $\theta = \tau_l$ $(l = 0, 1, \ldots, \nu,\ \vec{x}^{(\nu)} = \vec{x})$ in order as $\vec{q}_\Delta(\theta) \in R^{3n}$. In the same way we define the broken line paths $a_{\Lambda'\Delta}(\theta) \in R^{4N}$ on $[0,T]$ for $a_{\Lambda'}^{(0)},\ldots,a_{\Lambda'}^{(\nu-1)}$ and $a_{\Lambda'}$. We define $a_{\Lambda\Delta}(\theta) \in R^{8N}$ by means of (2.13). We write the classical action

$$\tilde{S}_c(T,0;\vec{q}_\Delta,a_{\Lambda\Delta}) = \int_0^T \tilde{\mathcal{L}}_c(\vec{q}_\Delta(\theta),\dot{\vec{q}}_\Delta(\theta),$$
$$a_{\Lambda\Delta}(\theta),\dot{a}_{\Lambda\Delta}(\theta))\,d\theta. \tag{3.5}$$

THEOREM 3.1. *We assume for $g(x)$ and $\psi(\theta)$ in* (3.4) *that for any $l = 1,2,\ldots$ and any multi-index α there exist constants $\delta_l > 0$ and $\delta_\alpha > 0$ satisfying*

$$|\partial_\theta^l \psi(\theta)| \le C_l(1+|\theta|)^{-(1+\delta_l)},\ \theta \in R$$

and

$$|\partial_x^\alpha g(x)| \le C_\alpha(1+|x|)^{-(1+\delta_\alpha)},\ x \in R^3,$$

respectively. Let $f(\vec{x},a_{\Lambda'}) \in L^2(R^{3n+4N})$. We write

$$\left(\prod_{j=1}^{n}\prod_{l=1}^{\nu}\sqrt{\frac{m_j}{2\pi i h(\tau_l-\tau_{l-1})}}^{\,3}\right)\prod_{l=1}^{\nu}\sqrt{\frac{1}{2|V|\pi i h(\tau_l-\tau_{l-1})}}^{\,4N}$$
$$\times\ Os - \int\int \left(\exp i h^{-1}\tilde{S}_c(T,0;\vec{q}_\Delta,a_{\Lambda\Delta})\right) f\Big(\vec{q}_\Delta(0),$$
$$a_{\Lambda'\Delta}(0)\Big)d\vec{x}^{(0)}\cdots d\vec{x}^{(\nu-1)}da_{\Lambda'}^{(0)}\cdots da_{\Lambda'}^{(\nu-1)} \tag{3.6}$$

as $(C_\Delta(T,0)f)(\vec{x},a_{\Lambda'})$ or $\iint \left(\exp i h^{-1}\tilde{S}_c(T,0;\vec{q}_\Delta,a_{\Lambda\Delta})\right) f(\vec{q}_\Delta(0),a_{\Lambda'\Delta}(0))$ $\times\mathcal{D}\vec{q}_\Delta\mathcal{D}a_{\Lambda'\Delta}$. Then, as $|\Delta|$ tends to 0, the function $(C_\Delta(T,0)f)(\vec{x},a_{\Lambda'})$ converges to the so-called Feynman path integral $\iint \left(\exp i h^{-1}\tilde{S}_c(T,0;\vec{q},a_\Lambda\right) f(\vec{q}(0),a_{\Lambda'}(0))\mathcal{D}\vec{q}\mathcal{D}a_{\Lambda'}$ in $L^2(R^{3n+4N})$. In addition, this limit satisfies the Schrödinger type equation

$$i h\frac{\partial}{\partial t}u(t) = H(t)u(t),\ u(0) = f, \tag{3.7}$$

where

$$
\begin{aligned}
H(t) = \sum_{j=1}^{n} \frac{1}{2m_j} \left| \frac{h}{i} \frac{\partial}{\partial x^{(j)}} - \frac{e_j}{c} \tilde{A}(x^{(j)}, a_\Lambda) \right|^2 \\
+ \frac{2\pi}{|V|} \sum_{j \neq l}^{n} \sum_{k \in \Lambda} \frac{e_j e_l \cos k \cdot (x^{(j)} - x^{(l)})}{|k|^2} \\
+ \sum_{k \in \Lambda', \mathrm{i}, \mathrm{l}} \left\{ \frac{|V|}{2} \left(\frac{h}{i} \frac{\partial}{\partial a_{\mathrm{l}k}^{(\mathrm{i})}} \right)^2 + \frac{(c|k|)^2}{2|V|} \left| a_{\mathrm{l}k}^{(\mathrm{i})} \right|^2 - \frac{hc|k|}{2} \right\}.
\end{aligned}
\tag{3.8}
$$

Remark 3.1. We suppose $\Lambda_2' \subseteq \Lambda_3'$. Then the same assertion as in Theorem 3.1 holds.

Remark 3.2. We note about the second term in (3.8) that we have

$$
\begin{aligned}
&\lim_{L_1, L_2, L_3 \to \infty} \lim_{M_1 \to \infty} \frac{2\pi}{|V|} \sum_{j \neq l}^{n} \sum_{k \in \Lambda_1} \frac{e_j e_l \cos k \cdot (x^{(j)} - x^{(l)})}{|k|^2} \\
&= \frac{1}{2} \sum_{j \neq l}^{n} \frac{e_j e_l}{|x^{(j)} - x^{(l)}|} \quad \text{in } \mathcal{S}'(R^{3n})
\end{aligned}
\tag{3.9}
$$

as in [2] and [5] by means of

$$
\frac{1}{(2\pi)^2} \int e^{ik \cdot x} / |k|^2 dk = \frac{1}{2} \frac{1}{|x|} \quad \text{in } \mathcal{S}'(R^3).
$$

Remark 3.3. In many literatures (cf. [6], [7], [13] and [14]) the Hamiltonian operator $H(t)$ defined by (3.8), the momentum operator and etc. are given by means of the creation operators and the annihilation operators. In this remark we give the expressions of $H(t)$, the momentum operator and etc. by means of the creation operators and the annihilation operators.

Let's define

$$\hat{a}_{lk}^{(i)} := i\sqrt{\frac{|V|}{2hc|k|}}\left(\frac{h}{i}\frac{\partial}{\partial a_{lk}^{(i)}} - i\frac{c|k|}{|V|}a_{lk}^{(i)}\right)$$
$$= \sqrt{\frac{|V|}{2hc|k|}}\left(h\frac{\partial}{\partial a_{lk}^{(i)}} + \frac{c|k|}{|V|}a_{lk}^{(i)}\right) \tag{3.10}$$

and

$$\hat{a}_{lk} := \frac{\hat{a}_{lk}^{(1)} - i\hat{a}_{lk}^{(2)}}{\sqrt{2}}. \tag{3.11}$$

Then, we call $\hat{a}_{lk}$ the annihilation operators and their adjoint operators $\hat{a}_{lk}^{\dagger}$ the creation operators. The operators $\hat{a}_{lk}$ and $\hat{a}_{lk}^{\dagger}$ sastisfy the commutator relations well known (cf. (2.26) in [13]). We can write the last term of $H(t)$ defined by (3.8) as

$$H_{\text{rad}} := \frac{1}{2}\sum_{k\in\Lambda,l}\sum_{i=1}^{2}\left\{\frac{|V|}{2}\left(\frac{h}{i}\frac{\partial}{\partial a_{lk}^{(i)}}\right)^2 + \frac{(c|k|)^2}{2|V|}\left|a_{lk}^{(i)}\right|^2 - \frac{hc|k|}{2}\right\}$$
$$= \sum_{k\in\Lambda,l} hc|k|\hat{a}_{lk}^{\dagger}\hat{a}_{lk}. \tag{3.12}$$

The vector potential $A(x,\{a_{lk}\})$ defined by (2.9) or (2.14), where the sum of k is taken over Λ_2, is given by the expression

$$A(x,\{a_{lk}\}) = \sqrt{\frac{4\pi h}{|V|}}c\sum_{k\in\Lambda_2}\sum_{l=1}^{2}\frac{1}{\sqrt{2c|k|}}\left(\hat{a}_{lk}e^{ik\cdot x} + \hat{a}_{lk}^{\dagger}e^{-ik\cdot x}\right)\vec{e_l}(k). \tag{3.13}$$

We see that

$$\Psi_0 := \prod_{k\in\Lambda',l}\frac{c|k|}{2\pi h|V|}\exp\left\{-\frac{c|k|}{2h|V|}\left(a_{lk}^{(1)^2} + a_{lk}^{(2)^2}\right)\right\}$$

is the ground state of H_{rad}, called vacuum, whose energy is 0, i.e.

$$H_{\text{rad}}\Psi_0 = 0$$

and satisfies

$$\hat{a}_{lk}^{\dagger}\Psi_0 = \sqrt{\frac{2c|k|}{h|V|}}a_{lk}^{*}\Psi_0, \ \hat{a}_{lk}\Psi_0 = 0 \tag{3.14}$$

(cf. Problem 9-8 in [5]). The function $\Psi_{n'lk} := (\hat{a}_{lk}^{\dagger})^{n'}\Psi_0$ ($n' = 1, 2, \dots$), which can be written in the concrete from (3.10) and (3.11), called n' photons with the momentum hk and the polarization state l (cf. [13]), satisfies

$$\left(\sum_{k\in\Lambda,l} \hat{a}_{lk}^{\dagger}\hat{a}_{lk}\right)\Psi_{n'l'k'} = n'\Psi_{n'l'k'}, \tag{3.15}$$

$$\left(\sum_{k\in\Lambda} hk\hat{a}_{lk}^{\dagger}\hat{a}_{lk}\right)\Psi_{n'l'k'} = n'(hk')\Psi_{n'l'k'} \tag{3.16}$$

and

$$H_{\text{rad}}\Psi_{n'l'k'} = n'(hc|k'|)\Psi_{n'l'k'} \tag{3.17}$$

from (3.14) and the commutation relations. We note that we assumed $\int A dx = 0$, i.e. $a_{l0}^{(i)} = 0$ (i, l $= 1, 2$) in (2.6). The operators defined by the left hand side of (3.15) and (3.16) are called the number operator and the momentum operator, respectively (cf. [13]).

Remark 3.4. In many literaturs (cf. [5], [13] and [14]) an arbitrary constant in the Lagrangian function (2.3) is determined to be 0. Consequently, the term $(1/2)\sum_{j=1}^{n} e_j^2/|x^{(j)} - x^{(j)}|$ appears in (3.9) and the ground state energy of H_{rad} is $\sum_{k\in\Lambda} hc|k|/2$, which tends to infinity when M_3 tends to infinity. In the present paper we determined an arbitrary constant in (2.3) by (2.18). Consequently, we could see that the term $(1/2)\sum_{j=1}^{n} e_j^2/|x^{(j)} - x^{(j)}|$ disappears in (3.9) and that the ground state energy of H_{rad} becomes 0.

Remark 3.5. We consider the external electromagnetic field $E_{\text{ex}}(t,x) = (E_{\text{ex}1}, E_{\text{ex}2}, E_{\text{ex}3}) \in R^3$ and $B_{\text{ex}}(t,x) = (B_{\text{ex}1}, B_{\text{ex}2}, B_{\text{ex}3}) \in R^3$. We assume as in Ichinose [8] that for any $\alpha \neq 0$ there exist constants C_α and $\delta_\alpha > 0$ satisfying

$$|\partial_x^\alpha E_{\text{ex}j}(t,x)| \le C_\alpha,\ |\alpha| \ge 1,\ |\partial_x^\alpha B_{\text{ex}j}(t,x)| \le C_\alpha(1+|x|)^{-(1+\delta_\alpha)}$$

(j = 1, 2, 3) in $[0,T] \times R^n$. Let $\phi_{\text{ex}}(t,x) \in R$ and $A_{\text{ex}}(t,x) \in R^3$ be the electromagnetic potential to E_{ex} and B_{ex}. We replace $\tilde{A}(x,\{a_{lk}\})$ in (3.3) and (3.8) with $\tilde{A}(x,\{a_{lk}\}) + \sum_{j=1}^n A_{\text{ex}}(t,x^{(j)})$. Moreover we add $-\sum_{j=1}^n e_j\phi_{\text{ex}}(t,x^{(j)})$ to (3.3) and $\sum_{j=1}^n e_j\phi_{\text{ex}}(t,x^{(j)})$ to (3.8), respectively. Then, the same assertion as in Theorem 3.1 holds.

<u>The outline of the proof of Theorem 3.1.</u> Let $\|f\|$ denote the L^2 norm for a function $f(\vec{x}, a_{\Lambda'})$ on R^{3n+4N}. For $a = 1,2,\dots$ we consider the weighted Sobolev spaces

$$\begin{aligned} B^a := \{f(\vec{x}, a_{\Lambda'}) \in L^2(R^{3n+4N});\ \|f\|_{B^a} := \|f\| + \\ \sum_{|\alpha|=a} (\|\boldsymbol{x}^\alpha f\| + \|(h\partial_{\boldsymbol{x}})^\alpha f\|) < \infty\},\ \boldsymbol{x} := (\vec{x}, a_{\Lambda'}). \end{aligned} \tag{3.18}$$

We set $B^0 = L^2$. Then, we can prove:

(1) There exist constants $\rho^* > 0$ and $K_a \ge 0$ $(a = 0,1,2,\dots)$ such that for $0 \le t \le T$ we have

$$\|\mathcal{C}_\Delta(t,0)f\|_{B^a} \le e^{K_a T}\|f\|_{B^a},\ 0 \le |\Delta| \le \rho^*, \tag{3.19}$$

where $\mathcal{C}_\Delta(t,0)f$ was defined by (3.6).

(2) There exists a constant $M \ge 2$ such that for $\le t, t' \le T$ and $a = 0,1,2,\dots$

we have

$$\left\| ih\left(\mathcal{C}_\Delta(t,0)f - \mathcal{C}_\Delta(t',0)f\right) - \int_{t'}^{t} H(\theta)\mathcal{C}_\Delta(\theta,0)f d\theta \right\|_{B^a}$$
$$\leq C_a\sqrt{|\Delta|}|t-t'|\|f\|_{B^{a+M}},\ 0 \leq |\Delta| \leq \rho^*, \tag{3.20}$$

where $H(t)$ is the Hamiltonian operator defined by (3.8).

We have

$$\|\mathcal{C}_\Delta(t,0)f - \mathcal{C}_\Delta(t',0)f\|_{B^a} \leq \text{Const.}|t-t'|\|f\|_{B^{a+M}} \tag{3.21}$$

from (3.19) and (3.20). It follows from the Rellich criterion (cf. [12]) that the embedding map from $B^{a+2M} \to B^{a+M}$ is compact. Let $f \in B^{a+2M}$. Then we can apply the abstract Ascoli-Arzerà theorem to $\{\mathcal{C}_\Delta(t,0)f\}_\Delta$ in $C^0([0,T];B^{a+M})$ from the compactness and the equicontinuity (3.21). Consequently, for any sequence $\{\Delta(n)\}_{n=1}^\infty$ such that $\lim_{n\to\infty}|\Delta(n)| = 0$, we can choose a subsequence $\{\Delta(n_j)\}_{j=1}^\infty$ such that there exists $\lim_{j\to\infty}\mathcal{C}_{\Delta(n_j)}(t,0)f$ uniformly in $C^0([0,T];B^{a+M})$. This limit satisfies the Schrödinger type equation (3.7) from (3.20). Hence, we can prove Theorem 3.1 by means of the uniqueness of solutions to (3.7), and (3.19) just above. See Ichinose [10] and [11] for details.

4 Results without the constraint condition.

In place of $\mathcal{L}$ expressed by (2.18) we consider

$$\tilde{\mathcal{L}}(\vec{x}, \dot{\vec{x}}, \{a_{lk}\}, \{\dot{a}_{lk}\}, \{\phi_k\}) := \sum_{j=1}^{n} \frac{m_j}{2}|\dot{x}^{(j)}|^2$$
$$+ \frac{1}{8\pi|V|} \sum_{k\in\Lambda_1} \left\{ \sum_{\mathrm{i}=1}^{2} \left(|k|^2 |\phi_k^{(\mathrm{i})}|^2 - 8\pi\rho_k^{(\mathrm{i})}(\vec{x})\, \phi_k^{(\mathrm{i})} \right) \right.$$
$$\left. + 16\pi^2 \frac{\sum_{j=1}^{n} e_j^2}{|k|^2} \right\} + \frac{1}{c}\sum_{j=1}^{n} e_j\, \dot{x}^{(j)} \cdot \tilde{A}(x^{(j)}, \{a_{lk}\})$$
$$+ \frac{1}{2} \sum_{k\in\Lambda_3,\mathrm{i},l} \left(\frac{|\dot{a}_{lk}^{(\mathrm{i})}|^2}{2|V|} - \frac{(c|k|)^2 |a_{lk}^{(\mathrm{i})}|^2}{2|V|} + \frac{hc|k|}{2} \right) \tag{4.1}$$

by means of (3.4) as in $\tilde{\mathcal{L}}_c$, where we suppose $\Lambda_2 \subseteq \Lambda_3$. For the sake of simplicity we again suppose $\Lambda' := \Lambda'_1 = \Lambda'_2 = \Lambda'_3$ as in Theorem 3.1.

Let $\vec{q}_\Delta(\theta) \in R^{3n}$, $a_{\Lambda'\Delta}(\theta) \in R^{4N}$ and $a_{\Lambda\Delta}(\theta) \in R^{8N}$ be the broken line paths defined before. Let $\vec{\xi_k} := \left\{\xi_k^{(\mathrm{i})}\right\}_{\mathrm{i}=1,2} \in R^2$ for $k \in \Lambda'$. Take $\vec{\xi_k}^{(0)}, \vec{\xi_k}^{(1)}, \ldots$ and $\vec{\xi_k}^{(\nu-1)}$ in R^2 arbitrarily. Let $\rho_k := (\rho_k^{(1)}, \rho_k^{(2)})$ from (2.16) and (2.17). Then, we define the path

$$\phi_{k\Delta}(\theta) := \vec{\xi_k}^{(l)} + \frac{4\pi\rho_k(\vec{q}_\Delta(\theta))}{|k|^2} \in R^2,\ \tau_{l-1} < \theta \le \tau_l \tag{4.2}$$

$(l = 1, 2, \ldots, \nu)$, where $\phi_{k\Delta}(0) := \lim_{\theta\to 0+0} \phi_{k\Delta}(\theta)$. We set $\phi_{\Lambda'\Delta}(\theta) := \{\phi_{k\Delta}(\theta)\}_{k\in\Lambda'}$ $\in R^{2N}$. We define $\phi_{\Lambda\Delta}(\theta) \in R^{4N}$ by means of (2.13). Let $\tilde{S}(T, 0; \vec{q}_\Delta, a_{\Lambda\Delta}, \phi_{\Lambda\Delta})$ be the classical action for $\tilde{\mathcal{L}}(\vec{x}, \dot{\vec{x}}, \{a_{lk}\}, \{\dot{a}_{lk}\}, \{\phi_k\})$.

THEOREM 4.1. *Let $f(\vec{x}, a_{\Lambda'}) \in B^a(R^{3n+4N})$ $(a = 0, 1, \ldots)$. Then as a* *function in* $B^a(R^{3n+4N})$ *we see that*

$$\left(\prod_{j=1}^{n}\prod_{l=1}^{\nu}\sqrt{\frac{m_j}{2\pi ih(\tau_l-\tau_{l-1})}}^{\,3}\right)\prod_{l=1}^{\nu}\left\{\prod_{k\in\Lambda'}\left(-\frac{i|k|^2(\tau_l-\tau_{l-1})}{4h\pi^2|V|}\right)\right.$$
$$\left.\times\sqrt{\frac{1}{2|V|\pi ih(\tau_l-\tau_{l-1})}}^{\,4N}\right\}Os-\int\cdots\int\Big(\exp ih^{-1}\tilde{S}(T,0;$$
$$\vec{q}_\Delta,a_{\Lambda\Delta},\phi_{\Lambda\Delta})\Big)f\left(\vec{q}_\Delta(0),a_{\Lambda'\Delta}(0)\right)d\vec{x}^{(0)}\cdots d\vec{x}^{(\nu-1)}$$
$$\times\, da_{\Lambda'}^{(0)}\cdots da_{\Lambda'}^{(\nu-1)}\prod_{k\in\Lambda'}d\vec{\xi_k}^{(0)}d\vec{\xi_k}^{(1)}\cdots d\vec{\xi_k}^{(\nu-1)} \tag{4.3}$$

is equal to

$$\iint\Big(\exp ih^{-1}\tilde{S}_c(T,0;\vec{q}_\Delta,a_{\Lambda\Delta})\Big)$$
$$\times\, f\left(\vec{q}_\Delta(0),a_{\Lambda'\Delta}(0)\right)\mathcal{D}\vec{q}_\Delta\mathcal{D}a_{\Lambda'\Delta}$$

defined by (3.6) *in Theorem 3.1. So it follows from Theorem 3.1 that as* $|\Delta|\to 0$, *then* (4.3) *converges to the Feynman path integral*

$$\iiint\Big(\exp ih^{-1}\tilde{S}(T,0;\vec{q},a_\Lambda,\phi_\Lambda)\Big)f\left(\vec{q}(0),a_{\Lambda'}(0)\right)\mathcal{D}\vec{q}\mathcal{D}a_{\Lambda'}\mathcal{D}\phi_{\Lambda'}, \tag{4.4}$$

which satisfies the Schrödinger type equation (3.7). *This expression* (4.4) *is given in §9-8 in Feynman-Hibbs [5], though a concrete definition is not given there.*

Remark 4.1. As was noted in the introduction, the constraint condition (2.19) isn't needed in Theorem 4.1 above.

Remark 4.2. We get the similar assertions for (4.3) as in Theorem 4.1 under the assumptions of Remark 3.1 and Remark 3.5, respectively.

<u>The outline of the proof.</u> Substitute (4.2) into $\tilde{\mathcal{L}}(\overrightarrow{q}_\Delta(\theta), \dot{\overrightarrow{q}}_\Delta(\theta), a_{\Lambda\Delta}(\theta), \dot{a}_{\Lambda\Delta}(\theta), \phi_{\Lambda\Delta}(\theta))$ and use

$$\sqrt{\frac{a}{i\pi}} \int_{-\infty}^{\infty} e^{iax^2} dx = 1 \ (a > 0).$$

Then we can prove Theorem 4.1 by means of the theory of the pseudo-differential operators. See Ichinose [9] and [11] for details.

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